

Network-Growth Rule Dependency of Percolation Criticality – Generalized Explosive Percolation –

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Abstract – To consider relation between percolation criticality and network-growth rule in an evolving network model, we propose a general network-growth rule which includes the conventional network-growth rule and the rule proposed by Achlioptas *et al.* very recently [Science **323** (2009) 1453]. We introduce the generalized parameter q which characterizes how to evolve a network. We obtain size dependency of percolation step and fractal dimension of percolated cluster at the percolation step for several q 's. Strong size dependency of percolation step and sudden change of fractal dimension are observed for intermediate q . From these facts, we find that the percolation criticality changes between at the conventional network-growth rule and at the rule proposed by Achlioptas *et al.*

Percolation transition has been a hot topic not only in statistical physics but also in a wide area of science [1–3]. In materials science, theory of percolation transition is used to investigate relation between connectivity and properties of materials such as electric conductivity in alloys, magnetic phase transition in diluted ferromagnets, and sol-gel transition in polymers. In natural science, percolation transition relates to dynamical features in nature such as spreading wildfire and spreading epidemic. Although these phenomena are non-linear non-equilibrium behaviors, we can consider these effects by just focusing on connectivity of composing elements in the network. In information science and engineering, theory of percolation transition can also adopt for dynamic nature of evolving network systems such as complex network. A curious example is to investigate relationship between percolation transition and PageRank which is an algorithm for deciding importance of website and is used in search engine [4]. Applicable scope of percolation theory has been extended day by day. On the other hand, it seems that nature of percolation transition such as relationship between the criticality and the spatial dimension has been well-established in terms of statistical physics.

In 2009, Achlioptas *et al.* proposed a novel type of

network-growth rule and studied nature of percolation transition [5]. In the conventional network-growth rule, we randomly select a bond between elements which do not belong to the same cluster and connect the bond definitely. On the other hand, in the rule proposed by Achlioptas *et al.*, we randomly select two bonds between elements where the elements adjoining each bond do not belong to the same cluster. After that we connect the bond where the product of sizes of two clusters which are contacted by the bond is smaller than the other. We refer to this rule as Achlioptas product rule. They calculated time evolution of the maximum size of clusters on system without spatial structure. Nature of time evolution resulting from the Achlioptas product rule is completely different from that obtained by the conventional network-growth rule. In fact, Achlioptas *et al.* claimed that a discontinuous phase transition takes place in network-growth model adopted the Achlioptas product rule, whereas a network-growth model using conventional rule exhibits a continuous phase transition. Since the size of the maximum clusters increases explosively against time in the Achlioptas product rule, the percolation transition is called explosive percolation transition. Some researchers have confirmed occurrence of explosive percolation transition [6, 7], whereas a num-

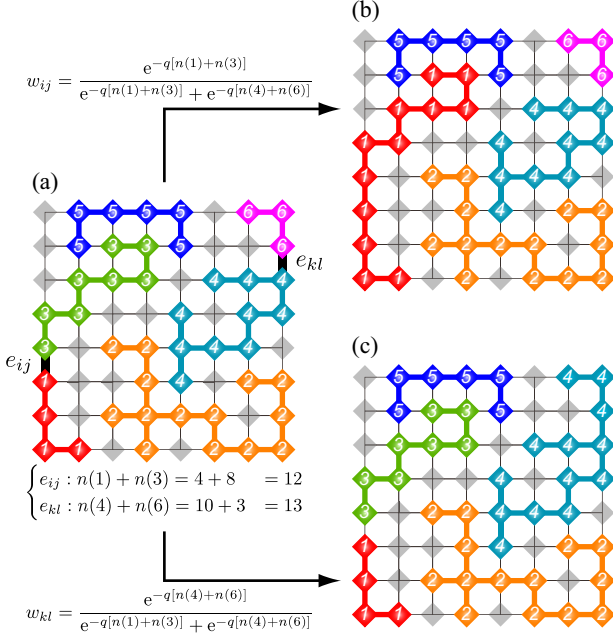


Fig. 1: (Color online) Graphical representation of our proposed rule on the square lattice. The numbers in the diamonds represent the state of vertices. The transparent gray diamonds denote the isolated clusters. (a) The bold lines denote randomly chosen edges e_{ij} and e_{kl} . In this case, the sums of the clusters are $n(1) + n(3) = 4 + 8 = 12$ and $n(4) + n(6) = 10 + 3 = 13$. We select $e_{ij}(e_{kl})$ as connecting edge with the probability $w_{ij}(w_{kl})$. In the random rule ($q = 0$), we select e_{ij} or e_{kl} with the same probability $1/2$. In the Achlioptas sum rule ($q = +\infty$), we definitely select e_{ij} whereas we select e_{kl} in the inverse Achlioptas sum rule ($q = -\infty$). (b) Configuration after the edge e_{ij} is selected and connected. (c) Configuration after the edge e_{kl} is selected and connected.

ber of researchers have insisted that the explosive percolation transition is actually continuous [8–12]. After this breakthrough study by Achlioptas *et al.*, a number of researchers have thoroughly investigated properties of explosive percolation transition on finite-dimensional lattices (regular lattices), scale-free networks, and random graphs by adopting the Achlioptas product rule [13–16]. Moreover, a couple of extended network-growth rules have been proposed and nature of percolation transitions using these rules have been studied [17–20]. The aim of these studies is to propose a rule which makes the percolation transition more explosively. In contrast, our purpose of this letter is to investigate relation between percolation criticality and network-growth rule. To study this problem in a clear manner, we adopt the sum rule as network-growth rule in this study. In the sum rule, we randomly select two bonds between elements which do not belong to the same cluster and connect a bond by comparing sums of the size of clusters whereas in most preceding studies, products of the size of clusters are compared. When we use

the sum rule, the number of elements obeys additivity between before connection of elements and after connection of elements because of the definition. In other words, the small cluster is definitely constructed in network-growth process based on the Achlioptas sum rule in which we connect the bond where the sum of sizes of two clusters is smaller than the other. Then we can figure out what happens clearly when we change how to evolve a network. The sum rule is the simplest and most appropriate rule to control the percolation phenomena such as the change of percolation step depending on the network-growth rule. To control a network-growth rule, we introduce a new parameter q which characterizes how to evolve a network. It should be noted that this is the first study where the sum rule is adopted for two-dimensional lattices. Our proposed rule includes the conventional rule and the Achlioptas sum rule as shown below.

Before we explain our proposed rule, we define network-growth model and notations in a general way. Let V and E be a set of vertices (points) and edges (sides), respectively. The i -th vertex is denoted by $v_i \in V$, and $e_{ij} \in E$ represents the edge between v_i and v_j . The state of the i -th vertex is represented by σ_i ($\sigma_i = \{1, \dots, N\}$), where N denotes the number of vertices. The state of the edge between the i -th and the j -th vertices is expressed by τ_{ij} ($\tau_{ij} = \{0, 1\}$). We assume that all edges are non-directed *i.e.* $\tau_{ij} = \tau_{ji}$ for $\forall i, j$. When the edge e_{ij} is not connected, $\tau_{ij} = 0$, whereas when the edge e_{ij} is connected, $\tau_{ij} = 1$. Cluster is defined as a set of vertices which have the same state. The number of elements in the cluster where v_i belongs to is expressed by $n(\sigma_i)$.

Here we explain our proposed network-growth rule. We show that our rule can describe the conventional network-growth rule which is referred to as “random rule” and the Achlioptas sum rule. The procedure of our proposed network-growth rule is as follows:

Step 1 The initial state is set to be $\sigma_i = i$ ($i = 1, \dots, N$) for $\forall i$, in other words, all elements belong to different clusters, *i.e.* $n(\sigma_i) = 1$ for $\forall i$ and $\tau_{ij} = 0$ for $\forall i, j$.

Step 2 We randomly choose two different edges $e_{ij} \neq e_{kl}$ satisfying conditions such that $\tau_{ij} = \tau_{kl} = 0$, $\sigma_i \neq \sigma_j$ and $\sigma_k \neq \sigma_l$.

Step 3 We connect e_{ij} with the probability w_{ij} defined by

$$w_{ij} := \frac{e^{-q[n(\sigma_i)+n(\sigma_j)]}}{e^{-q[n(\sigma_i)+n(\sigma_j)]} + e^{-q[n(\sigma_k)+n(\sigma_l)]}}, \quad (1)$$

whereas we connect e_{kl} with the probability $w_{kl} := 1 - w_{ij}$. After we connect $e_{ij}(e_{kl})$, the states of one of the clusters where $v_j(v_l)$ belongs to are changed to $\sigma_i(\sigma_k)$. When we connect an edge, time advances from T to $T + 1$.

Step 4 We repeat step 2 and step 3 until all of the elements belong to the same cluster.

Here we assume that all clusters are never separated. In this network-growth rule, the number of clusters decreases one by one in each time. In fact, the number of clusters at time T is $N - T$ for $0 \leq T \leq N - 1$. The graphical representation of the above procedure on the square lattice is summarized in Fig. 1.

The introduced parameter q in Eq. (1) is a generalized parameter and characterizes the network-growth rule. Our proposed rule for $q = 0$ is equivalent to the random rule as follows. In the random rule we randomly choose an edge e_{ij} such that $\sigma_i \neq \sigma_j$ and $\tau_{ij} = 0$ and connect the edge e_{ij} . In Eq. (1) for $q = 0$, the probabilities are the same $w_{ij} = w_{kl} = 1/2$, and thus it is equivalent to select the connecting edge randomly. On the other hand, our rule for $q = +\infty$ realizes the Achlioptas sum rule. In this rule, we randomly choose two edges e_{ij} and e_{kl} such that the conditions which stated in step 2 are satisfied. We compare the sums $n(\sigma_i) + n(\sigma_j)$ and $n(\sigma_k) + n(\sigma_l)$ and connect the edge where sum is smaller than the other. Actually, for $q = +\infty$ in Eq. (1), $w_{ij} = 1$ and $w_{kl} = 0$ when $n(\sigma_i) + n(\sigma_j) < n(\sigma_k) + n(\sigma_l)$ and vice versa. Hereafter we refer to our rule for $q = -\infty$ as “inverse Achlioptas sum rule”. In this case we select the connecting edge where the sum of elements is larger than the other. In this way, network-growth rule can be controlled by the generalized parameter q . Thus, we can explore relationship between network-growth rule and nature of percolation transition through the generalized parameter q . Notice that because of the conditions imposed in step 2, a cluster has no loop as well as “loopless” percolation [21]. This treatment is the same as the Achlioptas product rule process considered by Ziff [15].

To investigate the network-growth dependencies of nature of percolation transition, its criticality, and fractal dimension of the percolated cluster, we study the network-growth model on $L \times L (= N)$ two-dimensional square lattice with open boundary condition. The coordinate of v_i is represented by the position vector $\mathbf{r}_i := (x_i, y_i)$ for $1 \leq x_i, y_i \leq L$.

In order to understand q -dependency of network evolution quantitatively, we calculate the dynamics of the number of elements in the maximum cluster n_{\max} defined by $n_{\max} := \max\{n(\alpha) | 1 \leq \alpha \leq N\}$. Here $n(\alpha)$ represents the number of elements such that $\sigma_i = \alpha$ ($1 \leq i \leq N$). The value of n_{\max} characterizes the connectivity of network and is often calculated in studies of conventional percolation. As stated above, since we assume that clusters are never divided, n_{\max} monotonically increases with time. Figure 2 shows time development of the density of elements in the maximum cluster n_{\max}/N for $q = -\infty$ (inverse Achlioptas sum rule), -1 , -10^{-1} , -10^{-2} , -10^{-3} , -10^{-4} , 0 (random rule), 2.5×10^{-5} , 5×10^{-5} , 10^{-4} , 2×10^{-4} , 10^{-3} , 10^{-2} , 10^{-1} , and $+\infty$ (Achlioptas sum rule) on 256×256 square lattice. These results are obtained by averaging out the results calculated over 1024 samples. Here we define t as the normalized time $t := T/N$. The step of rise monotonically increases as q increases as shown

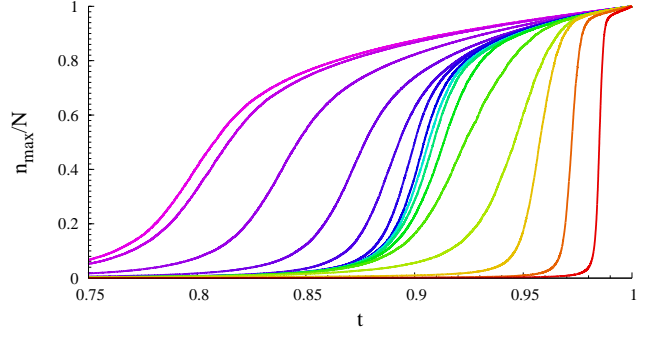


Fig. 2: (Color online) Time evolution of the density of elements in the maximum cluster n_{\max}/N on 256×256 square lattice for $q = -\infty$ (inverse Achlioptas sum rule), -1 , -10^{-1} , -10^{-2} , -10^{-3} , -10^{-4} , 0 (random rule), 2.5×10^{-5} , 5×10^{-5} , 10^{-4} , 2×10^{-4} , 10^{-3} , 10^{-2} , 10^{-1} , and $+\infty$ (Achlioptas sum rule) from left to right. These results are obtained by averaging out 1024 independent samples. Since the error bars are smaller than the widths of lines, they are omitted.

in Fig. 2. Thus, we can control network-growth speed by tuning network-growth rule. In contrast, the gradient at the step when n_{\max}/N reaches near $1/2$ behaves non-monotonically against the parameter q . This fact relates to the size dependency of the percolation step as explained later.

So far, we constructed a general network-growth rule by introducing the generalized parameter q and showed the q -dependency of dynamical nature of network evolution. Next we concentrate on the q -dependency of percolation point. In this study we define percolation as follows. If there is a cluster that spreads from left-side to right-side or from top to bottom, we call it percolated cluster. This is a typical definition in the percolation theory. Here we consider size dependency of the percolation step $t_p(L)$ for several q 's. The percolation step is defined by the step when a cluster becomes first percolated. The value of $t_p(L)$ is obtained by averaging out the obtained individual percolation step from 1024 samples. Figure 3 (a) shows size dependency of percolation step $t_p(L)$ for several q 's which are the same values using Fig. 2. As q increases, the percolation step increases monotonically. This fact is consistent with the result shown in Fig. 2. The percolation step $t_p(L)$ strongly depends on the lattice size around $q \simeq 10^{-4}$, which is the reason why the gradient of n_{\max}/N at the step when n_{\max}/N reaches near $1/2$ behaves non-monotonic against q in Fig. 2. Figure 3 (b) shows q -dependency of the percolation step for $L = 256$. At $q \simeq 10^{-4}$, this value suddenly increases as q increases. Next we study q -dependency of geometric aspect of the percolated cluster at the percolation step. To consider it, we calculate the ratio between the number of elements in the percolated cluster n_p and that of surface elements in the percolated cluster n_s . The surface element means the element that adjoins the other clusters or an isolated ele-

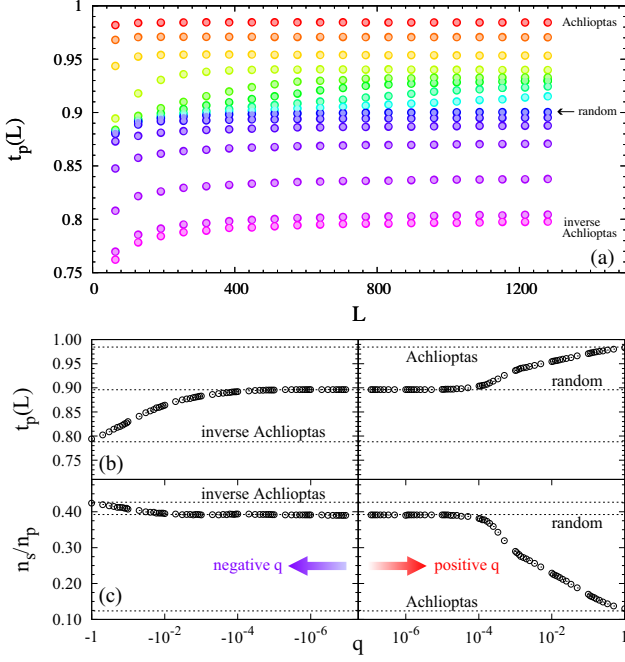


Fig. 3: (Color online) (a) Size dependency of the percolation step $t_p(L)$ for $q = -\infty$ (inverse Achlioptas sum rule), -1 , -10^{-1} , -10^{-2} , -10^{-3} , -10^{-4} , 0 (random rule), 2.5×10^{-5} , 5×10^{-5} , 10^{-4} , 2×10^{-4} , 10^{-3} , 10^{-2} , 10^{-1} , and $+\infty$ (Achlioptas sum rule) from bottom to top. (b) q -dependency of the percolation step $t_p(L)$ for $L = 256$. (c) q -dependency of the ratio n_s/n_p for $L = 256$. These results are obtained by averaging 1024 independent samples. Since the error bars of these results are smaller than the symbol sizes, they are omitted.

ment. Figure 3 (c) shows q -dependency of the ratio n_s/n_p for $L = 256$. At $q \simeq 10^{-4}$, this value suddenly decreases as q increases. In other words, geometry of the percolated cluster abruptly changes at near $q \simeq 10^{-4}$. These obtained results indicate that the percolation criticality around $q \simeq 10^{-4}$ is different from that of other regions including the random rule and the Achlioptas sum rule.

In order to investigate relation between the geometry of percolated cluster at the percolation step and the network-growth rule more quantitatively, we consider q -dependency of fractal dimension. The fractal dimension \mathcal{D} can be calculated by the radius gyration \mathcal{R} and the number of elements in the percolated cluster n_p . The radius gyration of the α -th cluster $\mathcal{R}(\alpha)$ is defined as

$$\mathcal{R}(\alpha) := \sqrt{\frac{1}{n(\alpha)} \sum_{i \text{ s.t. } \sigma_i = \alpha} |\mathbf{r}_i - \mathbf{r}_0(\alpha)|^2}, \quad (2)$$

$$\mathbf{r}_0(\alpha) := \frac{1}{n(\alpha)} \sum_{i \text{ s.t. } \sigma_i = \alpha} \mathbf{r}_i, \quad (3)$$

where the summation takes over vertices such that $\sigma_i = \alpha$ and $\mathbf{r}_0(\alpha)$ represents the position vector of the gravity point of the α -th cluster. In this study the definition of

the fractal dimension \mathcal{D} is adopted as follows:

$$n_p \propto \mathcal{R}_p^{\mathcal{D}}, \quad (4)$$

where \mathcal{R}_p denotes the radius gyration of the percolated cluster at the percolation step. The fractal dimension quantitatively characterizes geometric properties of fractal systems and is often used in analysis of the fractal geometries. The fractal dimension displays a criticality in percolation transition. When $\mathcal{D} = d$, where d is a spatial dimension, the geometry has no fractality. If $0 \leq \mathcal{D} \leq d$, on the other hand, the geometry of the percolated cluster has fractality as well as conventional self-similar structure. The upper panels of Fig. 4 show snapshots of the percolated cluster and those of the second-largest cluster for $q = -\infty$ (inverse Achlioptas sum rule), -10^{-2} , 0 (random rule), 10^{-5} , 10^{-2} , and $+\infty$ (Achlioptas sum rule) from left to right. The corresponding radius gyration dependency of the number of elements in the percolated cluster is shown in the lower panels of Fig. 4, which are obtained by calculation from $L = 64$ to $L = 1280$. The dotted lines are obtained by least-squares estimation using Eq. (4). Since these graphs are double logarithmic plots, the gradients of these fitting curves express the fractal dimension. The values of fractal dimensions are denoted in Fig. 4. The obtained fractal dimensions for all negative q are the same as that for $q = 0$ (random rule) $\mathcal{D} \simeq 1.88$ within the error bar. Moreover, for small $q (\leq 10^{-5})$, the fractal dimension is also the same as that for $q = 0$ within the error bar. It should be noted that the fractal dimension for this region is the same value as the conventional percolation transition on square lattice [3]. On the other hand, the fractal dimensions for positive value $q (\geq 10^{-2})$ are the same as that for the Achlioptas sum rule $\mathcal{D} \simeq 1.97$ within the error bar. The fractal dimensions of the random and Achlioptas sum rule are clearly distinguishable. Here we should notice that the fractal dimension for Achlioptas product rule is the same value as in the case of Achlioptas sum rule. Then qualitative similar behaviors are expected even when we adopt the rule based on product rule as most preceding studies. Thus we conclude there are at least two regions of q in terms of fractal dimension, and between at these regions the percolation criticality changes. It is an open problem to investigate the change of the fractal dimension in the intermediate q in detail.

In this study we proposed a novel network-growth rule in which the introduced parameter q assigns the rule. Since our rule includes the Achlioptas rule which was introduced in 2009 [5] and the conventional network-growth rule, our rule can be regarded as a general network-growth rule. We studied time evolution of the number of maximum cluster. We concentrated on the rule dependency of percolation criticality which is characterized by size-dependency of the percolation step and the fractal dimension. Strong size-dependency of percolation step in the case of $q \simeq 10^{-4}$ were observed. Fractal dimensions for several q were also calculated. The fractal dimensions at all negative q and the small positive value ($q \leq 10^{-5}$)

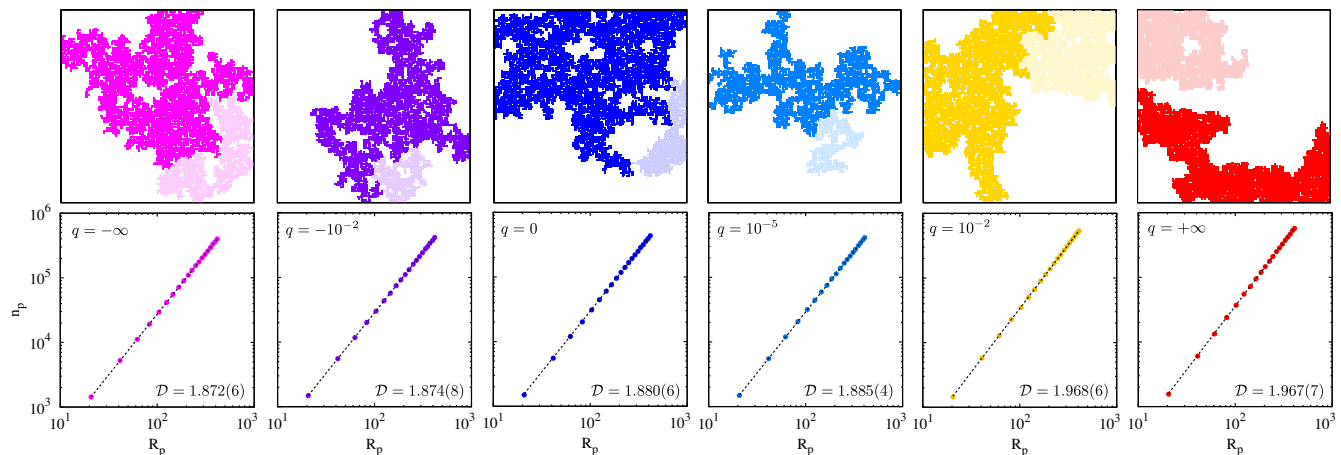


Fig. 4: (Color online) (Upper panels) Snapshots at the percolation step for $q = -\infty$ (inverse Achlioptas sum rule), -10^{-2} , 0 (random rule), 10^{-5} , 10^{-2} , and, $+\infty$ (Achlioptas sum rule). The dark and light colors depict the percolated cluster and the second-largest cluster, respectively. (Lower panels) Number of elements in the percolated cluster n_p as a function of radius gyration R_p for corresponding q . The dotted lines are obtained by least-squares estimation using Eq. (4) and the fractal dimensions are denoted.

are $\mathcal{D} \simeq 1.88$. On the other hand, the fractal dimensions for $10^{-2} \leq q \leq +\infty$ are $\mathcal{D} \simeq 1.97$. Thus, we conclude that the criticality of percolation transition changes around $q \simeq 10^{-4}$ by considering network-growth rule in full detail.

In this study we focused on the case for two-dimensional square lattice. To investigate the relation between the spatial dimension and the percolation criticality for our proposed rule is a remaining problem. Moreover, our rule is a general rule for many network-growth problem and enables us to design percolation criticality. Then we strongly believe that our rule will provide a greater understanding of percolation transition and will be applied for network-growth phenomena in nature and information technology.

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REFERENCES

- [1] KIRKPATRICK S., *Rev. Mod. Phys.*, **45** (1973) 574.
- [2] STAUFFER D., *Phys. Rep.*, **54** (1979) 1.
- [3] STAUFFER D. and AHARONY A., *Introduction to Percolation Theory* (Taylor & Francis, London) 1994.
- [4] FRAHM K. M., GEORGEOT B. and SHEPELYANSKY D. L., *J. Phys. A: Math. Theor.*, **44** (2011) 465101.

- [5] ACHLIOPTAS D., D’SOUZA R. M. and SPENCER J., *Science*, **323** (2009) 1453.
- [6] ARAÚJO N. A. M. and HERRMANN H. J., *Phys. Rev. Lett.*, **105** (2010) 035701.
- [7] CHO Y. S., KIM S.-W., NOH J. D., KAHNG B. and KIM D., *Phys. Rev. E*, **82** (2010) 042102.
- [8] DA COSTA R. A., DOROGOVSEV S. N., GOLTSEV A. V. and MENDES J. F. F., *Phys. Rev. Lett.*, **105** (2010) 255701.
- [9] RADICCHI F. and FORTUNATO S., *Phys. Rev. E*, **81** (2010) 036110.
- [10] GRASSBERGER P., CHRISTENSEN C., BIZHANI G., SON S.-W., PACZUSKI M., *Phys. Rev. Lett.*, **106** (2011) 225701.
- [11] LEE H. K., KIM B. J., and PARK H., *Phys. Rev. E*, **84** (2011) 020101(R).
- [12] RIORDAN O. and WARNKE L., *Science*, **333** (2011) 322.
- [13] FRIEDMAN E. J. and LANDSBERG A. S., *Phys. Rev. Lett.*, **103** (2009) 255701.
- [14] RADICCHI F. and FORTUNATO S., *Phys. Rev. Lett.*, **103** (2009) 168701.
- [15] ZIFF R. M., *Phys. Rev. Lett.*, **103** (2009) 045701.
- [16] ZIFF R. M., *Phys. Rev. E*, **82** (2010) 051105.
- [17] ANDRADE JR. J. S., HERRMANN H. J., MOREIRA A. A., and OLIVEIRA C. L. N., *Phys. Rev. E*, **83** (2011) 031133.
- [18] BASHAN A., PARSHANI R., and HAVLIN S., *Phys. Rev. E*, **83** (2011) 051127.
- [19] CHEN W. and D’SOUZA R. M., *Phys. Rev. Lett.*, **106** (2011) 115701.
- [20] NAGLER J., LEVINA A., and TIMME M., *Nature Physics*, **7** (2011) 265.
- [21] MANNA S. S. and SUBRAMANIAN B., *Phys. Rev. Lett.*, **76** (1999) 3460.